Feedforwrad Control Final Project
Inversion-Based Trajectory Control for 3-DoF Quadrotor

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Abstract

The purpose of this paper aims to design the feedforward control law of the three-degree-of-freedom (3-DoF) quadrotor for the trajectory tracking tasks. The main difficulties to seek the feedforward law of the quadrotor are that the relative degree is not well-defined and the nonlinearities of the quadrotor system. For these reasons, designing the feedforward law becomes hard. In this study, an effective approach for determining the feedforward control law is proposed. This approach allows the system to track the smooth reference trajectory, and it is expectable to realize the other nonlinear systems witch relative degree is not well-defined. To inhibit the effects of model uncertainties, the feedback-feedforward control scheme is also presented. The simulation results reveal that the system can successfully track the prespecified trajectory and indicate the effectiveness of the proposed control scheme.

Keywords: 3-DoF quadrotor, trajectory tracking, feedforward control, feedback-feedforward control.

1 Introduction

In recent years, due to its simple dynamic configuration and easy hardware implementation, quadrotors have been widely used in various fields, such as transportation, investigation, search and rescue, and so on. And there are several studies focus on control of quadrotor. For instance, a integral

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backstepping controller [1], minimum-time trajectory generation [2], body-rate control [3], optimal translational control [4], and dynamic inversion [5].

In this research, we devote ourselves to designing the feedforward control law to realize the trajectory tracking control of the 3-DoF quadrotor. In Section 2, the dynamic configuration and the governing equations are introduced. Based on the dynamic property, in Section 3, the feedforward control law is proposed, and the trajectory of the inverse system is solved. In Section 4, to inhibit the effects of model uncertainties, the feedback-feedforward control scheme is presented. In Section 5, the comparative simulations are performed to illustrate the effectiveness of the proposed feedforward controller and feedback-feedforward controller. Finally, the conclusion and future work are made in Section 6.

2 Dynamics Modeling

Consider the 3-DoF quadrotor dynamics in the X-Z plane. The dynamic configuration is shown in Figure 1. In which, $O(X,Y,Z)$ and $G(x,y,z)$ represent the global frame and the body-fixed frame, respectively. The distance between the rotor and the center of mass G is ℓ. The mass is m and g is the gravitational constant. The net force F_z and net torque M_y are generated by the thrusts of the two rotors, F_1 and F_2. Based on geometric analysis, we have

$$\begin{aligned} F_z &= F_1 + F_2 \\ M_y &= (F_1 - F_2)\,\ell \end{aligned} \tag{1}$$

It can be represented as a matrix form

$$\begin{bmatrix} F_z \\ M_y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \ell & -\ell \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \tag{2}$$

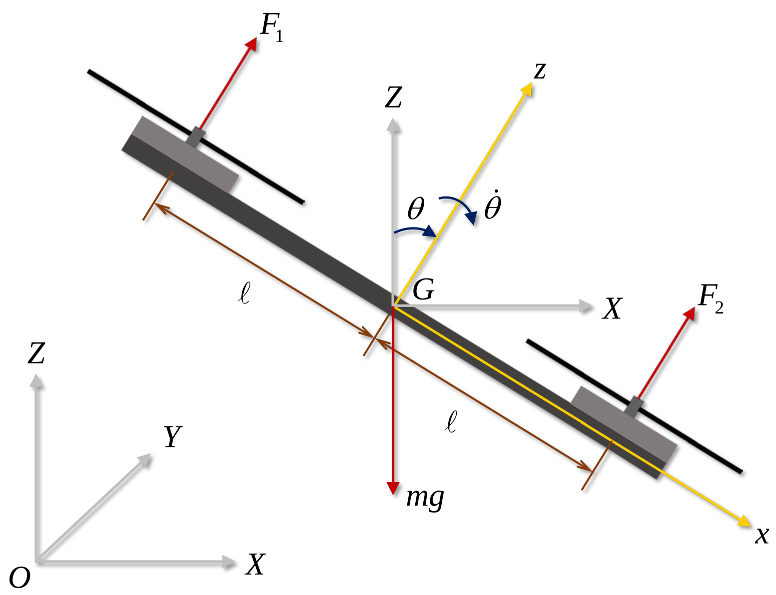


Figure 1: Dynamic configuration of 3-DoF quadrotor.

Let (X, Z) be the position and θ be the attitude of the body. Applying the Newton's second law and Euler equation of motion, the dynamic equations of 3-DoF quadrotor are given by

$$\begin{aligned} \ddot{X} &= \sin\theta \, \frac{F_z}{m} \\ \ddot{Z} &= \cos\theta \, \frac{F_z}{m} - g \\ \ddot{\theta} &= \frac{1}{J_y} M_y \end{aligned} \tag{3}$$

where J_y is the moment of inertia.

For the purpose of control design, define the state variables as

$$\begin{aligned} x_1 &= X \\ x_2 &= \dot{X} \\ x_3 &= Z \\ x_4 &= \dot{Z} \\ x_5 &= \theta \\ x_6 &= \dot{\theta} \end{aligned} \tag{4}$$

the corresponding state-space equations are given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \sin x_5 \, \frac{u_1}{m} \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \cos x_5 \, \frac{u_1}{m} - g \\ \dot{x}_5 &= x_6 \\ \dot{x}_6 &= \frac{1}{J_y} u_2 \end{aligned} \tag{5}$$

where $u_1 = F_z$ and $u_2 = M_y$ are the control inputs. Once the control inputs (u_1, u_2) are designed, the corresponding desired thrusts (F_1, F_2) can be obtained by the inverse mapping of (2):

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2\ell} \\ \frac{1}{2} & -\frac{1}{2\ell} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{6}$$

3 Feedforward Controller Design

The feedforward controller aims to determine a control law that drives the system to the prespecified reference trajectory and it doesn't involve any feedback info. Thus, the control objectives should be firstly defined. Let $y_1 = X = x_1$ and $y_2 = Z = x_3$ be the control objectives to be tracked.

Differentiating y_1 and y_2 until u_1 and u_2 appear:

$$\begin{aligned}
\dot{y}_1 &= \dot{x}_1 = x_2 \\
\ddot{y}_1 &= \dot{x}_2 = \sin x_5 \frac{u_1}{m} \qquad (7) \\
\dot{y}_2 &= \dot{x}_3 = x_4 \\
\ddot{y}_2 &= \dot{x}_4 = -g + \cos x_5 \frac{u_1}{m} \qquad (8)
\end{aligned}$$

Eq. (7) and (8) can be represented as a matrix form

$$\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -g \end{bmatrix} + \frac{1}{m} \begin{bmatrix} \sin x_5 & 0 \\ \cos x_5 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad (9)$$

Clearly, the relative degree of (u_1, u_2) to (y_1, y_2) is not well-defined since the matrix

$$\begin{bmatrix} \sin x_5 & 0 \\ \cos x_5 & 0 \end{bmatrix} \qquad (10)$$

is rank deficient. Therefore, we have to seek another approach to acquire the direct relation of (u_1, u_2) to (y_1, y_2).

First of all, consider (8) to design the feedforwrad control law of u_1 as

$$u_{1,ff} = \frac{m}{\cos x_5} \left(y_{2d}^{(2)} + g \right) \qquad (11)$$

where $y_{2d}^{(2)} = Z_d^{(2)}$. The control law $u_{1,ff}$ drive the dynamics (8) to the reference trajectory $y_{2d} = Z_d$. And then, substituting (11) into $\ddot{y}_1$ gives

$$\ddot{y}_1 = \tan x_5 \left(y_{2d}^{(2)} + g \right) \qquad (12)$$

Therefore, the dynamics (12) does not appear any control. It can continue to differentiate $\ddot{y}_1$ until u_2 appears:

$$\begin{aligned}
y_1^{(3)} &= \frac{d \tan x_5}{d x_5} \cdot \dot{x}_5 \cdot \left(y_{2d}^{(2)} + g \right) + \tan x_5 \cdot y_{2d}^{(3)} \\
&= a_1 x_6 \left(y_{2d}^{(2)} + g \right) + a_0 y_{2d}^{(3)} \qquad (13) \\
y_1^{(4)} &= \frac{d a_1}{d x_5} \cdot \dot{x}_5 \cdot x_6 \cdot \left(y_{2d}^{(2)} + g \right) + a_1 \cdot \ddot{\theta} \cdot \left(y_{2d}^{(2)} + g \right) + a_1 \cdot x_6 \cdot y_{2d}^{(3)} \\
&\quad + \frac{d a_0}{d x_5} \cdot \dot{x}_5 \cdot y_{2d}^{(3)} + a_0 \cdot y_{2d}^{(4)} \\
&= a_2 x_6^2 \left(y_{2d}^{(2)} + g \right) + \frac{1}{J_y} a_1 \left(y_{2d}^{(2)} + g \right) u_2 + 2 a_1 x_6 y_{2d}^{(3)} + a_0 y_{2d}^{(4)} \\
&\triangleq \alpha + \beta u_2 \qquad (14)
\end{aligned}$$

where

$$
\begin{aligned}
a_0 &= \tan x_5 \\
a_1 &= \frac{da_0}{dx_5} = \frac{d\tan x_5}{dx_5} = \sec^2 x_5 \\
a_2 &= \frac{da_1}{dx_5} = \frac{d^2\tan x_5}{dx_5^2} = 2\tan x_5 \sec^2 x_5 = 2a_0a_1
\end{aligned}
\tag{15}
$$

and

$$
\begin{aligned}
\alpha &= a_2 x_6^2 \left(y_{2d}^{(2)} + g\right) + 2a_1 x_6 y_{2d}^{(3)} + a_0 y_{2d}^{(4)} \\
\beta &= \frac{1}{J_y} a_1 \left(y_{2d}^{(2)} + g\right)
\end{aligned}
\tag{16}
$$

Finally, the feedforward control law of u_2 can be designed as

$$
u_{2,ff} = \beta^{-1}\left(y_{1d}^{(4)} - \alpha\right) \tag{17}
$$

The control law $u_{2,ff}$ drive the dynamics (14) to the reference trajectory $y_{1d} = X_d$.

Substituting $u_{1,ff}$ and $u_{2,ff}$ into the system dynamics (5) produces the following inverse system:

$$
\begin{aligned}
\dot{x}_{1,ref} &= x_{2,ref} \\
\dot{x}_{2,ref} &= \sin x_{5,ref} \frac{u_{1,ff}}{m} \\
\dot{x}_{3,ref} &= x_{4,ref} \\
\dot{x}_{4,ref} &= \cos x_{5,ref} \frac{u_{1,ref}}{m} - g \\
\dot{x}_{5,ref} &= x_{6,ref} \\
\dot{x}_{6,ref} &= \frac{1}{J_y} u_{2,ff} \\
u_{1,ff} &= \frac{m}{\cos x_{5,ref}} \left(y_{2d}^{(2)} + g\right) \\
u_{2,ff} &= \beta^{-1}\left(y_{1d}^{(4)} - \alpha\right)
\end{aligned}
\tag{18}
$$

where α and β can be obtained by replacing x_5 and x_6 with $x_{5,ref}$ and $x_{6,ref}$ in (16), respectively.

By off-line integrating the inverse system (18), the reference states $\mathbf{x}_{ref} = [x_{1,ref}, \cdots, x_{6,ref}]^T$ can be solved and the feedforward control inputs $u_{1,ff}$ and $u_{ff,2}$ are then constructed. In fact, the solution of $x_{1,ref}$, $x_{2,ref}$, $x_{3,ref}$, and $x_{4,ref}$ are the reference trajectories y_{1d}, $\dot{y}_{1d}$, y_{2d}, and $\dot{y}_{2d}$, respectively. Thus, we can directly replace these reference states by the reference trajectories to avoid integration errors. Furthermore, observing (12) and (13), one has

$$
\begin{aligned}
y_{1d}^{(2)} &= \tan x_{5,ref} \left(y_{2d}^{(2)} + g\right) \\
y_{1d}^{(3)} &= a_1 x_{6,ref} \left(y_{2d}^{(2)} + g\right) + a_0 y_{2d}^{(3)}
\end{aligned}
\tag{19}
$$

Thus, the reference states $x_{5,ref}$ and $x_{6,ref}$ can be acquired by the inverse kinematics of (19):

$$\begin{aligned} x_{5,ref} &= \tan^{-1}\left(\frac{y_{1d}^{(2)}}{y_{2d}^{(2)}+g}\right) \\ x_{6,ref} &= \frac{y_{1d}^{(3)} - a_0 y_{2d}^{(3)}}{a_1\left(y_{2d}^{(2)}+g\right)} \end{aligned} \tag{20}$$

Notice that a_0 and a_1 are obtained by replacing x_5 with $x_{5,ref}$ in (15).

4 Feedback-Feedforward Controller

From the practical realization point of view, the modeling errors/uncertainties will lead to the reference trajectory can not be achieved by the feedforward inputs. Thus, we are going to design the feedback controller to inhibit the effects of uncertainties. A simple and effective approach, local linearization, is performed to obtain a state feedback controller for the nonlinear system.

Consider a class of autonomous nonlinear system

$$\dot{\mathbf{x}}(t) = \mathbf{f}\left(\mathbf{x}(t), \mathbf{u}(t)\right), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{21}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector; $\mathbf{u}(t) \in \mathbb{R}^m$ is the control vector.

Let the nominal trajectory $(\mathbf{x}_n, \mathbf{u}_n)$ is governed by the following dynamics:

$$\dot{\mathbf{x}}_n(t) = \mathbf{f}\left(\mathbf{x}_n(t), \mathbf{u}_n(t)\right), \quad \mathbf{x}_n(0) = \mathbf{x}_{n0} \tag{22}$$

where $\mathbf{x}_n(t) \in \mathbb{R}^n$ is the nominal state vector and $\mathbf{u}_n(t) \in \mathbb{R}^m$ is the nominal control vector.

Suppose that $\mathbf{f}(\mathbf{x}, \mathbf{u})$ is analytical at $\mathbf{x} = \mathbf{x}_n$. Taking the Taylor series expansion of $\mathbf{f}\left(\mathbf{x}(t), \mathbf{u}(t)\right)$ about $(\mathbf{x}_n, \mathbf{u}_n)$ yields

$$\mathbf{f}\left(\mathbf{x}, \mathbf{u}\right) = \mathbf{f}\left(\mathbf{x}_n, \mathbf{u}_n\right) + \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{(\mathbf{x}_n, \mathbf{u}_n)} \Delta\mathbf{x} + \left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_{(\mathbf{x}_n, \mathbf{u}_n)} \Delta\mathbf{u} + \mathcal{O}\left(\|\Delta\mathbf{x}\|^2, \|\Delta\mathbf{u}\|^2\right) \tag{23}$$

where $\Delta\mathbf{x} = \mathbf{x} - \mathbf{x}_n$ and $\Delta\mathbf{u} = \mathbf{u} - \mathbf{u}_n$ are the deviation of system trajectory (22) and nominal trajectory (22); and $\mathcal{O}\left(\|\Delta\mathbf{x}\|^2, \|\Delta\mathbf{u}\|^2\right)$ represent the higher-order expansion terms. Combining (22), (22) and (23) produces

$$\Delta\dot{\mathbf{x}} = \mathbf{A}^*\Delta\mathbf{x} + \mathbf{B}^*\Delta\mathbf{u} + \mathcal{O}\left(\|\Delta\mathbf{x}\|^2, \|\Delta\mathbf{u}\|^2\right) \tag{24}$$

where $\Delta\dot{\mathbf{x}} = \dot{\mathbf{x}} - \dot{\mathbf{x}}_n$; the linearized system matrix $\mathbf{A}^* \in \mathbb{R}^{n\times n}$ and linearized input matrix $\mathbf{B} \in \mathbb{R}^{n\times m}$ are defined by

$$\begin{aligned} \mathbf{A}^* &= \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{(\mathbf{x}_n, \mathbf{u}_n)} \\ \mathbf{B}^* &= \left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_{(\mathbf{x}_n, \mathbf{u}_n)} \end{aligned} \tag{25}$$

Now we in view of system dynamics (5). Select the inverse system (18) as the nominal trajectory. Linearizing (5) about $(\mathbf{x}_{ref}, \mathbf{u}_{ff})$ gives

$$\dot{\mathbf{x}} - \dot{\mathbf{x}}_{ref} = \mathbf{A}^* \left(\mathbf{x} - \mathbf{x}_{ref}\right) + \mathbf{B}^* \left(\mathbf{u} - \mathbf{u}_{ff}\right) + \mathcal{O}\left(\|\Delta \mathbf{x}\|^2, \|\Delta \mathbf{u}\|^2\right) \tag{26}$$

where $\Delta\mathbf{x} = \mathbf{x} - \mathbf{x}_{ref}$, $\Delta\mathbf{u} = \mathbf{u} - \mathbf{u}_{ff}$ and the system matrices are

$$\begin{aligned}\mathbf{A}^* &= \left.\frac{\partial \mathbf{f}(\mathbf{x}, \mathbf{u})}{\partial \mathbf{x}}\right|_{(\mathbf{x},\mathbf{u})=(\mathbf{x}_{ref},\mathbf{u}_{ff})} \\ &= \left.\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{m}u_1 \cos x_5 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{m}u_1 \sin x_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right|_{(\mathbf{x},\mathbf{u})=(\mathbf{x}_{ref},\mathbf{u}_{ff})} \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{m}u_{1,ff} \cos x_{5,ref} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{m}u_{1,ff} \sin x_{5,ref} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\end{aligned} \tag{27}$$

and

$$\begin{aligned}\mathbf{B}^* &= \left.\frac{\partial \mathbf{f}(\mathbf{x}, \mathbf{u})}{\partial \mathbf{u}}\right|_{(\mathbf{x},\mathbf{u})=(\mathbf{x}_{ref},\mathbf{u}_{ff})} \\ &= \left.\begin{bmatrix} 0 & 0 \\ \frac{1}{m}\sin x_5 & 0 \\ 0 & 0 \\ \frac{1}{m}\cos x_5 & 0 \\ 0 & 0 \\ 0 & \frac{1}{J_y} \end{bmatrix}\right|_{(\mathbf{x},\mathbf{u})=(\mathbf{x}_{ref},\mathbf{u}_{ff})} \\ &= \begin{bmatrix} 0 & 0 \\ \frac{1}{m}\sin x_{5,ref} & 0 \\ 0 & 0 \\ \frac{1}{m}\cos x_{5,ref} & 0 \\ 0 & 0 \\ 0 & \frac{1}{J_y} \end{bmatrix}\end{aligned} \tag{28}$$

On the basis of linearized matrices $\mathbf{A}^*$ and $\mathbf{B}^*$, the feedback law is designed as

$$\mathbf{u} - \mathbf{u}_{ff} = -\mathbf{K}^* \left(\mathbf{x} - \mathbf{x}_{ref}\right) \tag{29}$$

where $\mathbf{K}^* \in \mathbb{R}^{2\times 6}$ is the state-feedback gain matrix which is designed so that the closed-loop system matrix

$$\mathbf{A}_c^* = \mathbf{A}^* - \mathbf{B}^*\mathbf{K}^* \tag{30}$$

is Hurwitz. Substituting (29) into (26), one has

$$\Delta\dot{\mathbf{x}} = \mathbf{A}_c^*\Delta\mathbf{x} + \mathbf{D}\left(\|\mathbf{x}\|^2\right) \tag{31}$$

where $\mathbf{D}\left(\|\mathbf{x}\|^2\right) = \mathcal{O}\left(\|\Delta\mathbf{x}\|^2, \|\mathbf{K}^*\Delta\mathbf{x}\|^2\right)$.

In what follows, we are going to investigate the stability of the closed-loop system (31). Assume that the perturbation term $\mathbf{D}\left(\|\mathbf{x}\|^2\right)$ satisfies

$$\left\|\mathbf{D}\left(\|\mathbf{x}\|^2\right)\right\| = \kappa(\mathbf{x})\|\mathbf{x}\| \le \kappa^+\|\mathbf{x}\| \tag{32}$$

Select the Lyapunov function candidate as

$$V = \Delta\mathbf{x}^T\mathbf{P}\Delta\mathbf{x} \tag{33}$$

where $\mathbf{P} = \mathbf{P}^T > 0$.

Taking the time derivative of V along the trajectory (31) yields

$$\dot{V} = \Delta\mathbf{x}^T\left(\mathbf{P}\mathbf{A}_c + \mathbf{A}_c^T\mathbf{P}\right)\Delta\mathbf{x} + 2\mathbf{D}^T\mathbf{P}\Delta\mathbf{x} \tag{34}$$

Since $\mathbf{A}_c^*$ is Hurwitz, there exists a positive definite matrix $\mathbf{Q} = \mathbf{Q}^T$ to satisfy the following Lyapunov equation:

$$\mathbf{P}\mathbf{A}_c + \mathbf{A}_c^T\mathbf{P} = \mathbf{Q} \tag{35}$$

Thus, $\dot{V}$ is then becomes

$$\dot{V} = -\Delta\mathbf{x}^T\mathbf{Q}\Delta\mathbf{x} + 2\Delta\mathbf{x}^T\mathbf{P}\mathbf{D} \tag{36}$$

Based on Rayleigh inequality, we have

$$-\lambda_{min}(\mathbf{Q})\|\Delta\mathbf{x}\|^2 \ge -\Delta\mathbf{x}^T\mathbf{Q}\Delta\mathbf{x} \ge -\lambda_{max}(\mathbf{Q})\|\Delta\mathbf{x}\|^2 \tag{37}$$

and

$$\begin{aligned}\Delta\mathbf{x}^T\mathbf{P}\mathbf{D} &\le \|\Delta\mathbf{x}\|\cdot\|\mathbf{P}\|\cdot\|\mathbf{D}\| \\ &= \lambda_{max}(\mathbf{P})\|\Delta\mathbf{x}\|\|\mathbf{D}\| \\ &\quad \text{[using (32)]} \\ &\le \kappa^+\lambda_{max}(\mathbf{P})\|\Delta\mathbf{x}\|^2\end{aligned} \tag{38}$$

Hence, (36) is governed by

$$\begin{aligned}\dot{V} &\le -\lambda_{min}(\mathbf{Q})\|\Delta\mathbf{x}\|^2 + 2\kappa^+\lambda_{max}(\mathbf{P})\|\Delta\mathbf{x}\|^2 \\ &= -\left(\lambda_{min}(\mathbf{Q}) - 2\kappa^+\lambda_{max}(\mathbf{P})\right)\|\mathbf{x}\|^2\end{aligned} \tag{39}$$

It can be concluded that if the upper bound κ^+ satisfies

$$\lambda_{min}(\mathbf{Q}) - 2\kappa^+ \lambda_{max}(\mathbf{P}) > 0 \quad \rightarrow \quad \kappa^+ < \frac{\lambda_{min}(\mathbf{Q})}{2\lambda_{max}(\mathbf{P})} \tag{40}$$

then it implies

$$\dot{V} < 0 \tag{41}$$

for all $\Delta\mathbf{x} \neq \mathbf{0}$ and $\dot{V} = 0$ as $\Delta\mathbf{x} = \mathbf{0}$. That is, the closed-loop system (31) is asymptotically stable, that is,

$$\lim_{t\to\infty} \mathbf{x} = \mathbf{x}_{ref} \tag{42}$$

Once the state-feedback gain $\mathbf{K}^*$ is desgined, from (29), it gives the overall feedback-feedforward control law:

$$\mathbf{u} = \mathbf{u}_{ff} - \mathbf{K}^* \left(\mathbf{x} - \mathbf{x}_{ref}\right) \tag{43}$$

Since the linearized system matrix $\mathbf{A}^*$ is varying with the reference trajectory $\left(\mathbf{x}_{ref}, \mathbf{u}_{ff}\right)$. The linear quadratic regulator (LQR) formulation is applied to design the state-feedback gain matrix $\mathbf{K}^*$. Consider the following infinite horizontal problem:

$$\min_{\Delta\mathbf{u}} \mathcal{J} = \int_0^\infty \Delta\mathbf{x}^T\mathbf{Q}\Delta\mathbf{x} + \Delta\mathbf{u}^T\mathbf{R}\Delta\mathbf{u}\, dt \tag{44}$$

subject to

$$\Delta\dot{\mathbf{x}} = \mathbf{A}^*\Delta\mathbf{x} + \mathbf{B}^*\Delta\mathbf{u}, \quad \Delta\mathbf{x}(0) = \Delta\mathbf{x}_0 \tag{45}$$

where $\mathbf{Q} > 0$ and $\mathbf{R} > 0$ both are appropriate dimension. It has been proven in several references that the solution of this problem is

$$\Delta\mathbf{u} = -\mathbf{K}^*\Delta\mathbf{x} = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\Delta\mathbf{x} \tag{46}$$

where $\mathbf{K}^* = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}$ is the steady-state Kalman gain matrix which can be obtained by solving the following algebraic Riccati equation (ARE):

$$\mathbf{PA}^* + \mathbf{A}^{*T}\mathbf{P} + \mathbf{PB}^*\mathbf{R}^{-1}\mathbf{B}^{*T}\mathbf{P} + \mathbf{Q} = \mathbf{0} \tag{47}$$

The design procedures for applying the feedback-feedforward controller are summarized as follows:

Step 1. Given a smooth reference trajectory $y_{1d}, \cdots, y_{1d}^{(4)}, y_{2d}, \cdots, y_{2d}^{(2)}$.

Step 2. Solve for the reference states from $x_{1,ref} = y_{1d}$, $x_{2,ref} = y_{1d}^{(1)}$, $x_{3,ref} = y_{2d}$, $x_{4,ref} = y_{2d}^{(1)}$; solve for $x_{5,ref}$, $x_{6,ref}$ from (20).

Step 3. Construct the feedforwad control inputs $u_{1,ff}$ and $u_{2,ff}$ from (11) and (17).

Step 4. Solve for $\mathbf{P}$ from the ARE (47) and compute the state-feedback gain $\mathbf{K}^* = \mathbf{R}^{-1}\mathbf{B}^{*T}\mathbf{P}$.

Step 5. The feedback-feedforwrad control law is acquired from (43) and use (6) to obtain the desired thrust inputs F_1, F_2.

5 Numerical Simulation

In this section, we consider the two case scenarios:

- Case A. The system without model uncertainty and applying the feedforward control only.

- Case B. The system with model uncertainties and using the feedback-feedforward control scheme. The first-order transfer function $G_a(s)$ is considered as the actuator dynamics describing the relation between the desired thrust inputs and actual thrust inputs.

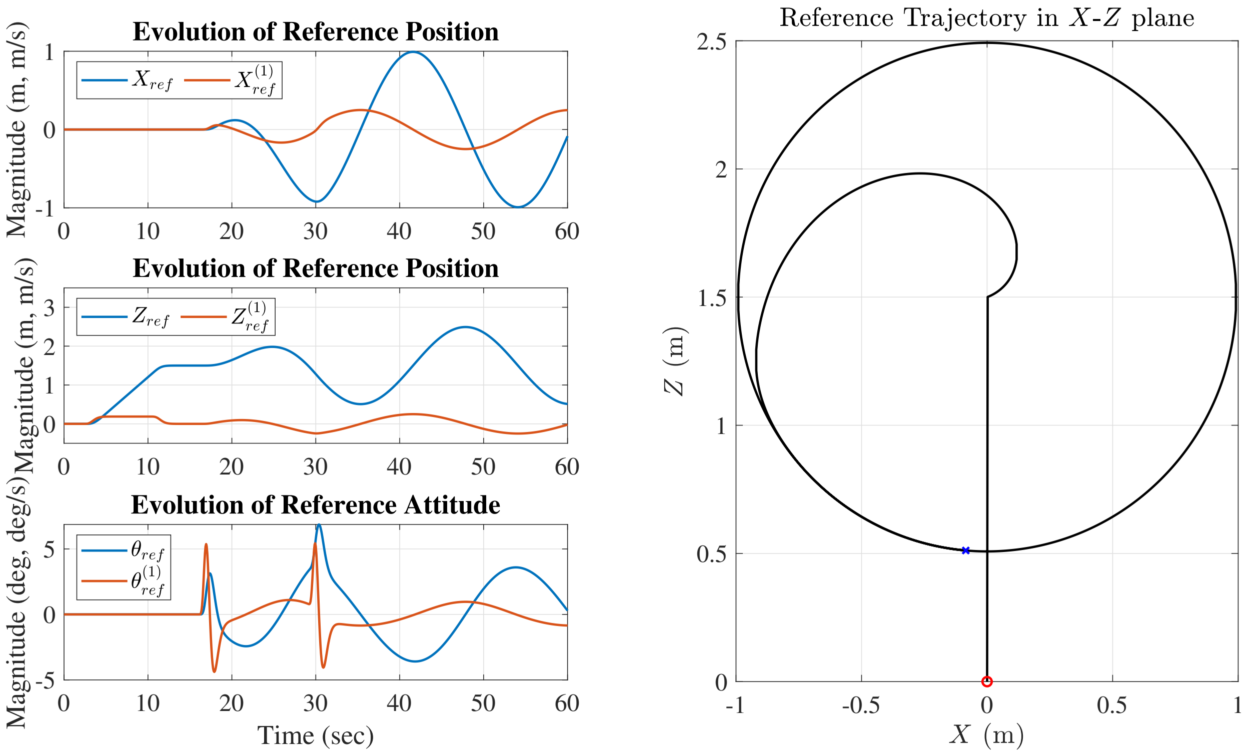
$$G_a(s) = \frac{T_2}{T_1 s + 1} \tag{48}$$

  where $T_1 = 0.001$ is the time constant and $T_2 = 0.995$ is the dc-gain.

The parameters m, g, J_y, ℓ are summarized as Table 1.

Table 1: The parameters of system.

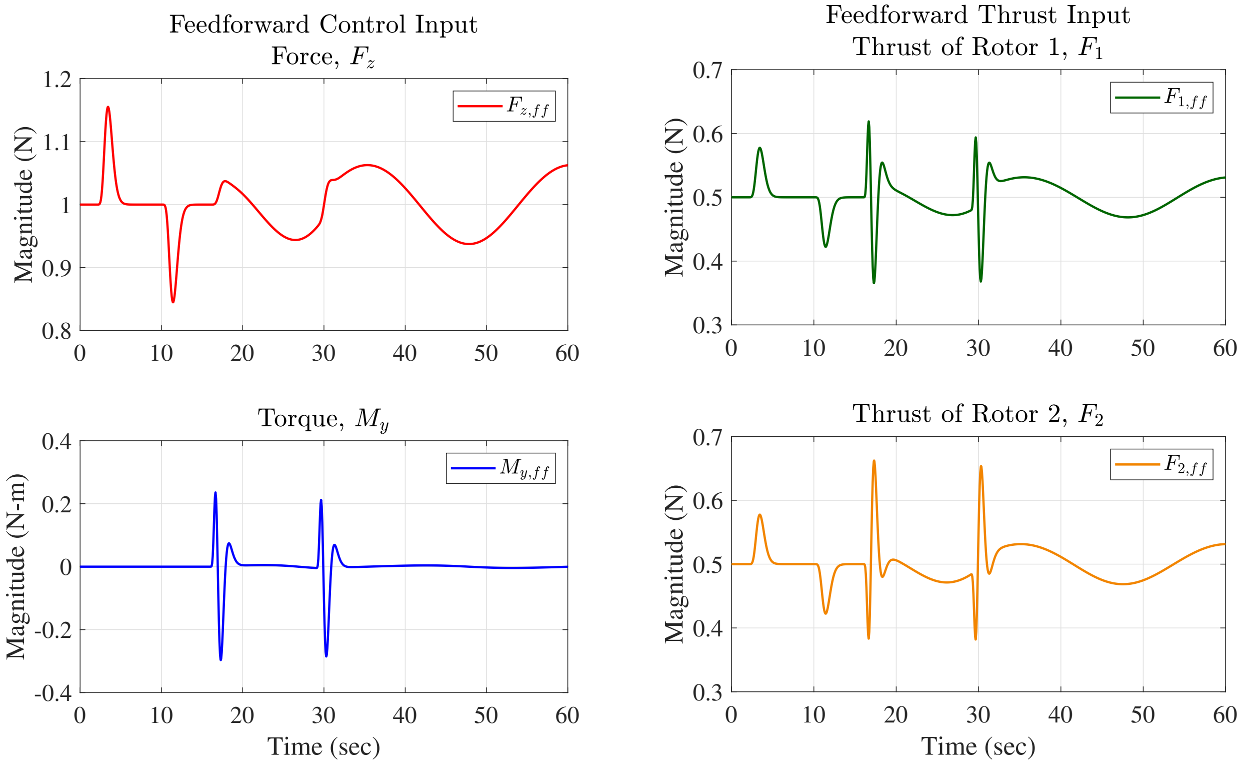
Quantity	m (kg)	g (m/s^2)	J_y (kg-m^2)	ℓ (m)
Nominal Value	1	1	1	1
Uncertainty	-0.5	0.05	0.7	0.01
Real Value	0.5	1.05	1.7	1.01



(a) Evolution of reference state.

(b) Reference Trajectory in X-Z plane

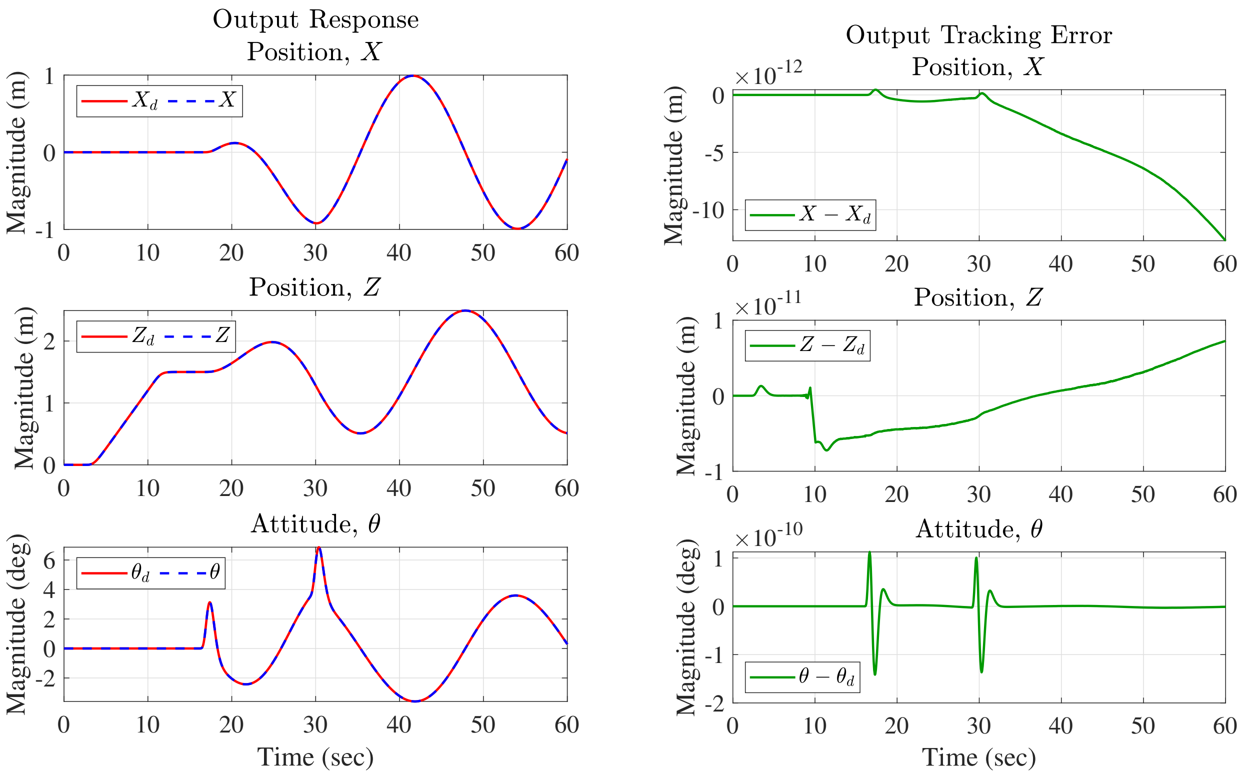
Figure 2: Evolution of reference states and the reference trajectory in X-Z plane.



(a) Feedforward control input.

(b) Feedforward thrust input.

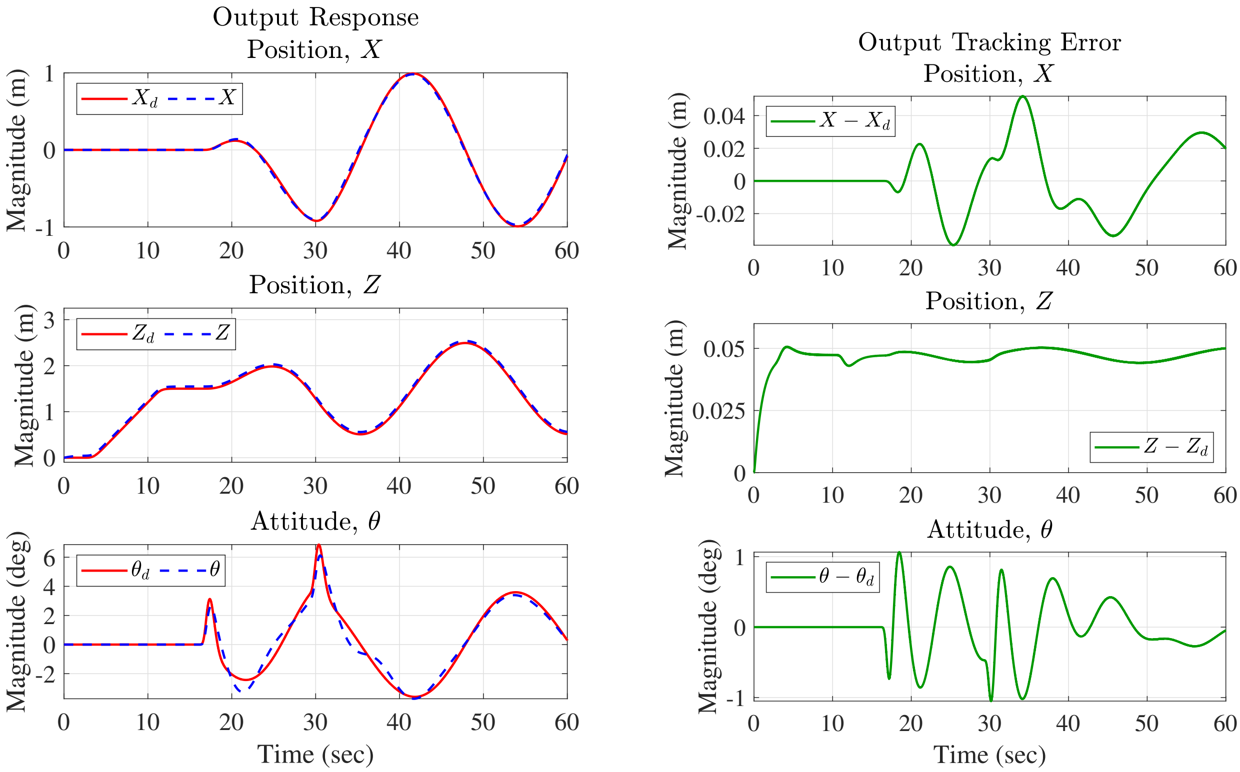
Figure 3: Feedforward control input and feedforward thrust input.



(a) Output response.

(b) Output tracking error.

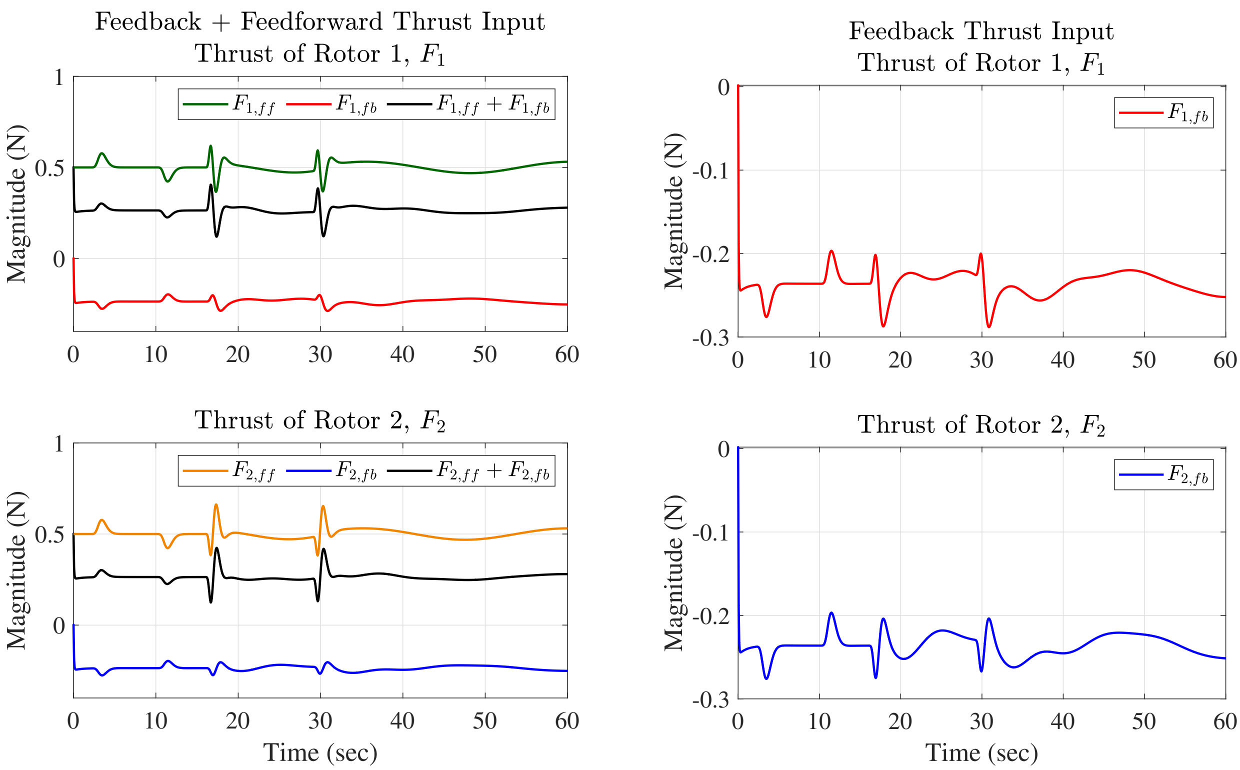
Figure 4: Output response and output tracking error (Case A).



(a) Output response.

(b) Output tracking error.

Figure 5: Output response and output tracking error (Csae B).



(a) Feedback and feedforwrad thrust input.

(b) Feedback input.

Figure 6: Comparison of feedback and feedforward thrust input (Case B).

Figure 2 shows that the reference states and the reference trajectory in X-Z plane. Figure 3 reveals the feedforward control inputs and the corresponding feedforward thrust inputs. Applying the feedforward thrust inputs to the system, the output response is shown in Figure 4. The result indicates that the system enables perfect tracking just using the feedforward inputs in the absence of the model uncertainty.

Figure 5 shows the output response of Case B. It can be found that the system still has a certain tracking performance under the influences of model uncertainties. The comparison of feedback and feedforward inputs are illustrated in Figure 6. It can be observed that the feedback inputs restrain the effects of model uncertainties and driving the system to the specified nominal trajectory as close as possible.

6 Conclusion

In this paper, the feedforward control law for the trajectory tracking of the 3-DoF quadrotor is proposed. The main difficulties are that the relative degree is not well-defined and the system nonlinearities. An effective approach is proposed to deal with these issues. The feedback-feedforward control scheme is presented to restrain the effects of the modeling error/uncertainties. The comparative simulations are performed, and the results reveal that the effectiveness of the proposed feedforward and feedback-feedforward control law.

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